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Construction of Second-Order TVD Schemes for Nonhomogeneous Hyperbolic Conservation Laws

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Many of the problems of approximating numerically solutions to nonhomogeneous hyperbolic conservation laws appear to arise from an inability to balance the source and flux terms at steady states. In this paper we present a technique based on the transformation of the nonhomogeneous problem to homogeneous form through the definition of a new flux formed by the physical flux and the primitive of the source term. This change preserves the mentioned balance directly and suggests a way to apply well-known schemes to nonhomogeneous conservation laws. However, the application of the numerical methods described for homogeneous conservation laws is not immediate and a new formalization of the classic schemes is required. Particularly, for such cases we extend the explicit, second-order, total variation diminishing schemes of Harten [11]. Numerical test cases in the context of the quasi-one-dimensional flow validate the current schemes, although these schemes are more general and can also be applied to solve other hyperbolic conservation laws with source terms. © 2001 Academic Press

1. INTRODUCTION

The present research is concerned with the formulation of conservative finite difference schemes with the total variation diminishing (TVD) property to solve systems of nonlinear hyperbolic conservation laws with source terms.

Nonhomogeneous systems of conservation laws arise naturally in many problems of practical interest. This includes, among others, Euler equations with a source term which has a geometric character. For example, the calculation of the unsteady one-dimensional flow in ducts of varying cross-sectional area as well as flow with cylindrical or spherical symmetry. Sources of similar types are present in the shallow water equations for flow on nonhorizontal channels.

Hyperbolic systems of conservation laws with a source term in one dimension can be written by the equation

$$W_t + F(W)_x = S(x, W), \tag{1}$$

where $W = W(x, t)$ is a m column vector formed by the flow variables, $F(W)$ is a vector-valued function of m components which includes the corresponding fluxes and $S(x, W)$ is the source vector.

In the past few years, a number of shock-capturing, finite difference schemes have been constructed to solve systems of homogeneous conservation laws, or which include an almost negligible source term (see, for example, [17] and [27]). These methods are characterized as being of second-order or by their higher accuracy in the smooth regions of the solution without presenting the spurious oscillations associated with the conventional second-order schemes in the presence of discontinuities. Harten in [11] introduced the TVD schemes, which have the property that they may be second-order accurate and oscillation-free across discontinuities. Special attention should be made to the proper formulation of those schemes when they are applied to nonhomogeneous hyperbolic systems. In this paper we are particularly concerned with the extension of the classic TVD schemes developed by Harten in [11] to nonhomogeneous conservation laws.

Until some years ago, several authors solved the nonhomogeneous problem by using a conservative finite difference method developed for the homogeneous system and then adding the effect of the source term only as a correction of the conservation step. This strategy is valid for certain types of problems, but when the source term has a strong influence on the solution, it becomes very inefficient and can lead to numerical errors, poor accuracy, and nonconvergence.

A commonly used technique to approximate solutions for nonhomogeneous conservation laws is the fractional step splitting method in which one alternates between solving, in each time step, the corresponding homogeneous system

$$W_t + F(W)_x = 0$$

and a system of ordinary differential equations

$$W_t = S(x, W)$$

allowing for the use of the optimal existing schemes for each subproblem. Particularly, we can use TVD (see [11, 12, and 25]), or ENO schemes (see [20] and [24]) to solve the homogeneous system. Nevertheless, while in some cases this approximation is quite good, in other cases the results are not suitable. In particular, these methods fail when the solution is close to a steady state in which the flux-gradient and the source terms should be compensated. In other words, these methods do not discretize the steady-state equation associated with (1)

$$F(W)_x = S(x, W).$$

Some authors, as Van Leer in [28], have already indicated the need to modify the numerical methods of the upwind type to solve single nonhomogeneous conservation laws substituting the initial numerical distribution that, for first-order methods, is considered uniform in each volume of control in the homogeneous case, by stationary distributions for the nonhomogeneous problem in each time step. This idea was also utilized in [10] by Glaz and Liu, who proposed a generalized Riemann problem where the initial data were not uniform on each cell but satisfied the steady-state equations so as to construct a random-choice method for quasi-one-dimensional flows.

Recently numerical schemes, based on flux discretizations which take the source terms into account (upwinding the source terms as well as the fluxes) or improve the resolution of the source terms, have been developed to solve nonhomogeneous hyperbolic conservation laws. In [23], Roe presented an empirical approach based on the application of upwind high-order schemes to a modified flux that includes the source terms. This method was applied by Glaister in [9] to solve the Euler equations of gas dynamics in one spatial coordinate.

In [26] Sweby reduced the nonhomogeneous problem to homogeneous form with a change of dependent variable; then TVD schemes can be effectively applied. This method includes the source terms without modifying the numerical flux, applying the TVD scheme only for the fluxes.

Leveque and Yee [16] utilized MacCormack-type predictor-corrector methods with flux limiters and splitting methods to incorporate the source terms, but the flux and the source terms were treated in separated steps in both cases.

Bermúdez and Vázquez [2] studied methods to get upwind discretizations of the source terms when the flux is approximated by using flux-difference or flux-vector splitting techniques. In order to find numerical schemes which approximate, exactly or with an order greater than one, stationary solutions for the shallow water equations, they introduced the conservation property and showed that the extensions of the Q-schemes of Van Leer and Roe verified this property but the extensions of the flux-vector splitting methods do not. In [29], Vázquez-Cendón generalizes these schemes for nonuniform meshes in order to solve the shallow water equations in channels with irregular geometry.

Motivated by the fact that if there is a source term the Riemann invariants are not constant along the characteristic trajectories, Papalexandris *et al.* [21] have described the curves in space–time along which the characteristic system holds for the nonhomogeneous case. This new decomposition is used by the authors in the design of efficient unsplit algorithms for the numerical integration of the systems of hyperbolic conservation laws with source terms.

A recent alternative approach has been introduced by Leveque [19], who has proposed the quasi-steady method based on the modification of the wave-propagation algorithms presented in [18] to achieve the balance between the flux and source terms for nonhomogeneous problems when the solution is close to a steady state. The balance is reached by introducing additional Riemann problems in the center of each grid cell whose flux difference cancels the source terms exactly. The same line has been adapted by the work of Jenny and Müller [15], who have introduced a new approach for a flux solver, the Rankine–Hugoniot–Riemann solver, which takes into account source terms, viscous terms, and multidimensional effects. The method is based on the transformation of the volume integral of the source terms into surface integrals.

In this paper, we describe a general method to extend the well-known TVD schemes introduced by Harten in [11] to hyperbolic conservation laws with source terms, giving rise to a set of sufficient conditions which are very useful in checking or constructing second-order TVD schemes for the nonhomogeneous case.

Our method is based on the following strategy. Suppose that we are interested in the steady-state solution associated with the problem

$$W_t + F(W)_x = S(x), \tag{2}$$

where the source term is independent of the conserved variable W. Integrating the stationary associated equation

$$F(W)_x = S(x)$$

the flux-vector $F(W)$ becomes

$$F(W) = K + \int_0^x S(y)\,dy \tag{3}$$

with K constant. This expression indicates that to secure a correct discretization of the stationary equation associated with nonhomogeneous conservation laws, it is convenient to describe schemes with the same treatment for the fluxes as for the primitive of the source terms; i.e., when the physical flow is upwinded the source terms also have to be upwinded and when one uses centered discretizations for the flux, one has to also use centered discretizations of source terms. Equation (3) suggests the form in which the source terms will be introduced into the methods. We shall denote

$$G(x, W) = F(W) - \int_0^x S(y)\,dy$$

and the original Eq. (2) can be written as

$$W_t + G(x, W)_x = 0, \tag{4}$$

where $G(x, W)$ is a new flux formed by the addition of the physical flux function and the primitive of the source term.

This transformation of the nonhomogeneous system in a homogeneous problem provides a suitable technique to apply TVD and other types of schemes, commonly used in the homogeneous case, to systems of conservation laws with source terms. Additionally, it allows us to include correctly the source terms as a divergence term providing the scheme the way to recognize steady solutions for nonhomogeneous conservation laws.

The present work can be seen as a formalization of the empirical technique suggested by Roe in [23] to limit the second-order terms in the nonhomogeneous case. As particular cases, the extensions of Q-schemes of van Leer and Roe to hyperbolic systems with source terms proposed by Bermúdez and Vázquez in [2] can be obtained directly by application to Eq. (4) of van Leer's classic Q-scheme and Roe's flux difference scheme, respectively, described originally for homogeneous hyperbolic systems in [13] and [22].

On the other hand, when the source term is considered dependent on W, new propagation speeds of the flow are introduced, which are the addition of the classic characteristic speeds introduced by the homogeneous case with a new term which is a consequence of the source terms' presence. This new contribution has the same form as the propagation speeds introduced by Papalexandris *et al.* in [21].

Before introducing the finite-difference formulation, we need to define our discretization nomenclature and indexing practices. We consider a fixed grid in space and time with grid sizes Δx and Δt, respectively. Integrating (1) on the rectangle $[x_{j-1/2}, x_{j+1/2}] \times [t_n, t_{n+1}]$, we obtain

$$W_j^{n+1} = W_j^n - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \left(F\left(W\left(x_{j+1/2}, t\right)\right) - F\left(W\left(x_{j-1/2}, t\right)\right)\right) dt \\ + \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} S(x, W(x,t))\, dx\, dt, \tag{5}$$

where the discrete nature of the problem forces us to replace the exact integrals by the average values for the variables W_j^n and W_j^{n+1}; i.e., for the numerical cell j, $[x_{j-1/2}, x_{j+1/2}]$, and in the time instant t_n, we denote

$$W_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} W(x, t_n)\, dx.$$

Equation (5) can be rewritten as

$$W_j^{n+1} = W_j^n - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \left(G\left(x_{j+1/2}, W\left(x_{j+1/2}, t\right)\right) - G\left(x_{j-1/2}, W\left(x_{j-1/2}, t\right)\right)\right) dt,$$

where

$$G(x, W) = F(W) - \int_0^x S(y, W(y,t))\, dy.$$

Then we shall discuss numerical approximations to solutions to (1) which are obtained by $(2k+1)$-explicit schemes written in conservation form

$$W_j^{n+1} = W_j^n - \lambda\left[\overline{G}_{j+1/2}^n - \overline{G}_{j-1/2}^n\right], \tag{6}$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and

$$\overline{G}_{j+1/2}^n = \overline{G}\left(x_{j-k+1}, \ldots, x_{j+k}, W_{j-k+1}^n, \ldots, W_{j+k}^n\right).$$

Here $\overline{G}$ is a numerical flux function constructed by the addition of a numerical flux function associated with the physical flow with a numerical source function corresponding to the primitive of the source term.

For consistency, we will assume

$$\overline{G}(x, \ldots, x, W, \ldots, W) = G(x, W).$$

If the numerical flux function associated with the physical flow is consistent with the physical flow $F(W)$—working with consistent schemes in the homogeneous case, for example—the above consistency property will only require that the discretization corresponding to the primitive of the source terms converging at the source term's primitive when Δt and Δx tend to 0.

Furthermore, in order to find suitable approximations to the steady-state solutions, we will require that the following discretized equation

$$\bar{G}^n_{j+1/2} - \bar{G}^n_{j-1/2} = 0 \tag{7}$$

be an approximation of at least a second-order to the stationary equation

$$F(W)_x = S(x, W)$$

because, if (W^n_j) approximates the stationary equation on the level n then, from (6), W^{n+1}_j and W^n_j will be equal and the scheme will recognize stationary solutions with at least second-order accuracy.

This paper is organized as follows. In Section 2, the development of an explicit second-order finite difference scheme based on the proposed strategy is presented as an extension of the classic Lax–Wendroff scheme for a scalar nonhomogeneous conservation law. This scheme does not prevent the total variation of the numerical approximations from increasing. For this, we study sufficient conditions in order to construct second-order TVD schemes for nonhomogeneous conservation scalar laws. Numerical experiments for the Embid problem (introduced in [7]) validate the results. The next section deals with the development of these ideas for the vectorial case. When the Jacobian matrix of the flux function and the evaluation of the source terms in the middle have been based on Roe's linearization technique we obtain, as a particular case, Roe's flux-difference scheme for conservation laws with source terms originally proposed in [23] and afterward in [2]. Similarly, we obtain the extension for nonhomogeneous conservation laws of the first-order Q-scheme of van Leer proposed in [2] when the arithmetic mean is applied to evaluate the Jacobian matrix. Finally, the application of the described schemes to solve Euler equations with source terms is explained and the calculation of results are presented in the last section. These results include examples for quasi-one-dimensional nozzle flows, which confirm that the method gives excellent results.

Further details about the description, analysis and application of the schemes described in this paper are presented in the Ph.D. thesis of the first author [8].

2. TVD, SECOND-ORDER ACCURATE SCHEMES FOR A CONSERVATION LAW WITH SOURCE TERMS

Classic schemes, modified slightly to take into account source terms, are very effective when they are applied to approximate numerically solutions to systems of conservation laws with a source term which has a low value with little influence on the solution. This is the case of 1-D flow with moderate friction and heat transfer in constant cross-section pipes. However, the authors found that those techniques failed in the calculation of high velocity flows where the variation of the cross-section of the pipe is from moderate to high. In this case, the source term includes terms related to the variation of pipe cross-sectional area and terms related to the presence of friction forces and heat transfer at pipe walls.

Traditionally, the two-step Lax–Wendroff method, corrected with a FCT technique (see [3]), has been largely chosen in fluid dynamics as a good compromise between accuracy and computational time for the calculation of unsteady flows in engine ducts with a constant cross-sectional area. This is the reason why we tried to apply the classic two-step Lax–Wendroff method to the governing equations of unsteady 1-D compressible flow through pipes with high ratios of cross-sectional variation. However, in these cases we found an important deviation from the exact solution.

In order to show the motivation for this work, we start this section illustrating the behavior of the two-step Lax–Wendroff scheme when applied to solve numerically nonhomogeneous conservation laws. Limiting our attention to the scalar case, we consider

$$w_t + f(w)_x = s(x, w), \tag{8}$$

where the source term is a smooth function of x and w.

This second-order explicit method taking into account the additional source term is based on two steps on a three-point stencil. First, $w_{j+1/2}^{n+1/2}$ is computed by the following equation

$$w_{j+1/2}^{n+1/2} = \frac{1}{2}\left[w_j^n + w_{j+1}^n - \lambda\left(f_{j+1}^n - f_j^n\right) + \frac{\Delta t}{2}\left(s_j^n + s_{j+1}^n\right)\right], \tag{9}$$

where $\lambda = \frac{\Delta t}{\Delta x}$. Then the solution w^{n+1} is evaluated by

$$w_j^{n+1} = w_j^n - \lambda\left[f_{j+1/2}^{n+1/2} - f_{j-1/2}^{n+1/2}\right] + \frac{\Delta t}{2}\left[s_{j+1/2}^{n+1/2} + s_{j-1/2}^{n+1/2}\right]. \tag{10}$$

In order to examine the capacity of the above second-order scheme to capture the steady states of nonhomogeneous conservation laws, we show numerical experiments for the Embid problem, described by the following nonlinear scalar conservation law with a source term explicitly dependent on x and w

$$\begin{cases} w_t + \left(\frac{1}{2}w^2\right)_x = (6x - 3)w, & 0 < x < 1 \\ w(0, t) = 1,\ w(1, t) = -0.1. \end{cases} \tag{11}$$

This problem was presented in [7] as a simple scalar approximation to the 1-D equations that model the flow of a gas through a duct of variable cross-section. It can be verified (see [7]) that there are two entropy satisfying steady solutions for the Embid problem (11). One is stable in time with a standing shock at $x_1 = 0.18$ and the other with an unstable standing shock at $x_2 = 0.82$. The steady solutions for the Embid problem (11) are

$$w(x) = \begin{cases} 1 + 3x^2 - 3x, & x < x_j \\ -0.1 + 3x^2 - 3x, & x > x_j \end{cases}$$

for $j = 1,\ 2$. We computed the steady profiles by taking initial data with a jump at the stable shock location.

Figure 1 shows the numerical solution for the Embid problem (11), calculated by the two-step, adapted Lax–Wendroff scheme compared with the exact solution (the solid line is the true solution). The computations were performed using 41 nodes equally spaced in the domain [0, 1] and the CFL equal to 0.25. The two-step Lax–Wendroff method was unstable and Fig. 1 shows numerical results obtained with this scheme after 800 iterations. We note that using other larger CFL numbers increases the instability of the method.

The numerical results obtained with this scheme for the present problem indicate that this method will not be a good candidate for the integration of nonhomogeneous conservation law systems, and particularly for the calculation of unsteady flows in engine ducts with variable cross-sectional area, because it does not recognize stationary solutions.

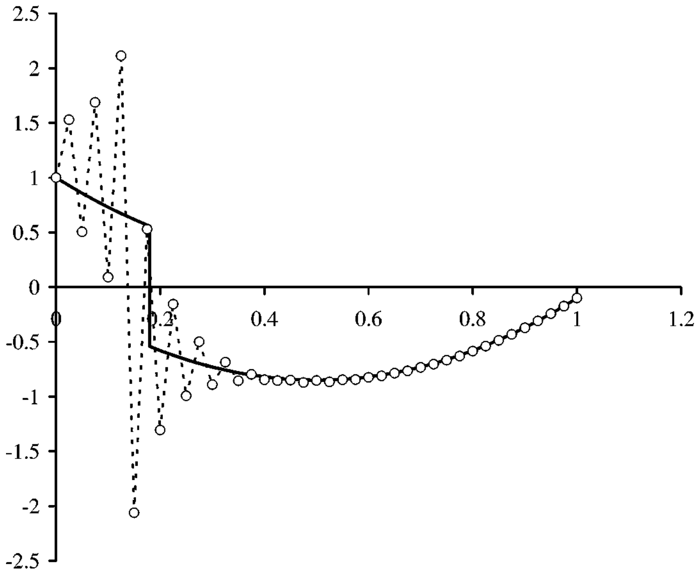

FIG. 1. Steady numerical solution for the Embid problem, calculated with the modified, two-step Lax–Wendroff scheme and compared with the exact solution (solid line).

We can state that using s_j^n as a discretization to the source term in the second stage of the two-step Lax–Wendroff method leads to identical numerical results.

We now see how we can define a second-order scheme for (8) following the strategy described in the Introduction. First, we propose to rewrite (8) using the following homogeneous law

$$w_t + g(x, w)_x = 0 \tag{12}$$

by introducing the flux function defined as

$$g(x, w) = f(w) - \int_0^x s(y, w(y, t))\, dy \tag{13}$$

and we propose an explicit three-point finite difference scheme in conservation form for the conservation law (12)

$$w_j^{n+1} = w_j^n - \lambda\left[g_{j+1/2}^{n+1/2} - g_{j-1/2}^{n+1/2}\right], \tag{14}$$

where the estimation of the new flux, g, at the point mid-way between grid points is obtained by an expansion formula based on Taylor series which takes the form of

$$g_{j+1/2}^{n+1/2} = \frac{1}{2}\left[g_j^n + g_{j+1}^n - \lambda \left.\frac{\partial g}{\partial w}\right|_{j+1/2}^{n} \left(g_{j+1}^n - g_j^n\right)\right]. \tag{15}$$

Here

$$g_i^n = f_i^n - \int_0^{x_i} s(y, w(y, t_n))\, dy \quad \text{for } i = j,\ j+1$$

and

$$\frac{\partial g}{\partial w} = \frac{\partial f}{\partial w} + \frac{\partial}{\partial w}\left(-\int_0^x s(y, w)\,dy\right).$$

The above scheme extends to systems in a similar way and defines a second-order accurate method, an extension of the one-step Lax–Wendroff scheme for nonhomogeneous conservation laws [6]. We can state that (14), (15) are obtained applying the classic one-step Lax–Wendroff scheme for homogeneous conservation laws to (12) and the presented scheme is reduced to the classic one-step Lax–Wendroff method for the particular case in which $g = f$, i.e., for the homogeneous case.

Furthermore, since

$$g^n_{j+1} - g^n_j = 0, \quad \forall j$$

is a second-order discretization to the stationary equation associated with (12), the scheme (14)–(15) recognizes stationary solutions for the nonhomogeneous case with the same accuracy.

By introducing the following notation

$$b^n_{i,k} = -\int_{x_i}^{x_k} s(y, w(y, t_n))\,dy$$

and using simple algebraic manipulations, we showed in [6] that the proposed scheme admits the expression

$$\begin{aligned} w^{n+1}_j = w^n_j &- \lambda\left[f^{LW}_{j+1/2} - f^{LW}_{j-1/2}\right] - \lambda\left[b^n_{j-1/2,j} + b^n_{j,j+1/2}\right] \\ &- \frac{\Delta t^2}{4\Delta x}\left[\left.\frac{\partial s}{\partial w}\right|^n_{j+1/2}\left(f^n_{j+1} - f^n_j + b^n_{j,j+1}\right) + \left.\frac{\partial s}{\partial w}\right|^n_{j-1/2}\left(f^n_j - f^n_{j-1} + b^n_{j-1,j}\right)\right], \end{aligned} \tag{16}$$

where

$$f^{LW}_{j+1/2} = \frac{1}{2}\left[f^n_j + f^n_{j+1} - b^n_{j,j+1/2} + b^n_{j+1/2,j+1} - \lambda\left.\frac{\partial f}{\partial w}\right|^n_{j+1/2}\left(f^n_{j+1} - f^n_j + b^n_{j,j+1}\right)\right]. \tag{17}$$

Figure 2 shows the numerical solution for the Embid problem calculated by the adapted, one-step Lax–Wendroff method, described by (16) and (17), compared with the exact solution (solid line). The steady solutions were calculated with this scheme using a CFL number equal to 0.8 and by marching in time until the convergence criterion

$$\max_j \left|w^{n+1}_j - w^n_j\right| \le 10^{-10}$$

was satisfied. We chose, in this case, the following approximations for $b^n_{i,k}$

$$b^n_{j-1/2,j} = -\frac{w^n_{j-1} + w^n_j}{2}\left[3x^2 - 3x\right]^{x_j}_{x_{j-1/2}}$$

$$b^n_{j,j+1/2} = -\frac{w^n_j + w^n_{j+1}}{2}\left[3x^2 - 3x\right]^{x_{j+1/2}}_{x_j}$$

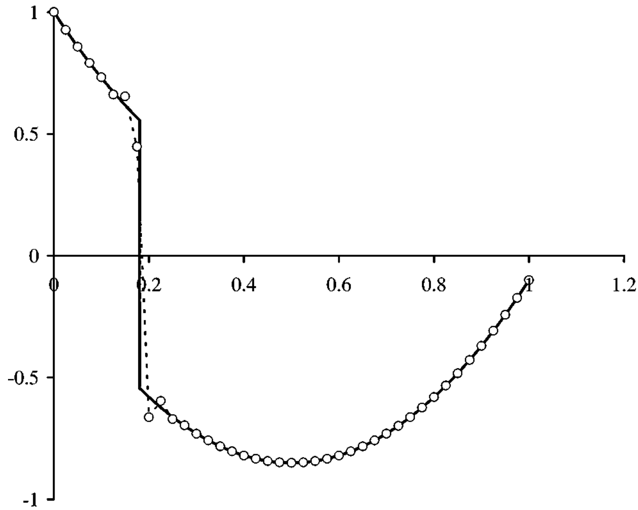

FIG. 2. Steady numerical solution for the Embid problem, calculated with the adapted, one-step Lax–Wendroff method and compared with the exact solution (solid line).

because then

$$f_{j+1} - f_j + b_{j,j+1} = 0 \Leftrightarrow \frac{w_{j+1}^2 - w_j^2}{2} = \frac{w_j + w_{j+1}}{2} \int_{x_j}^{x_{j+1}} (6x - 3)\, dx$$
$$\Leftrightarrow w_{j+1} - w_j = \int_{x_j}^{x_{j+1}} (6x - 3)\, dx$$

and the last equation is exact for all smooth solutions satisfying the steady equation associated with the Embid problem (11)

$$w_x = (6x - 3);$$

then we can conclude that the scheme (14)–(15) recognizes exactly stationary solutions for the Embid problem in smooth regions.

Numerical results with other choices for $b_{j,j+1}^n$ has been omitted because the differences would not be visible in the graphs. However, in the particular case of the Embid problem this approximation proved to be the most accurate. It is worthwhile noting the similarity of our evaluation for $b_{j,j+1}^n$ with the proposed approximation of the source term employed in [2] for the Saint–Venant equations in order to satisfy the *exact $\mathcal{C}$-property*.

Although the accuracy of the solution calculated with the adapted, one-step Lax–Wendroff scheme is very good in smooth regions, the inevitable presence of spurious overshoots in the proximity of the shock, typical of second-order schemes, has been observed. This motivates the need to build TVD schemes to compute solutions for nonhomogeneous conservation laws maintaining the balance between flux and source terms at steady states. To this end, we need to construct efficient techniques with a limitation of the second-order terms and with the capacity to recognize stationary solutions for conservation laws with source terms.

As opposed to the homogeneous case, where the classic TVD schemes introduced by Harten to integrate the homogeneous equation

$$w_t + f(w)_x = 0 \tag{18}$$

have the form

$$w_j^{n+1} = w_j^n - \lambda\left[\tilde{f}_{j+1/2} - \tilde{f}_{j-1/2}\right],$$

where the evaluation of the flux in the middle between two points is given by

$$\tilde{f}_{j+1/2} = \frac{1}{2}\left[f_j^n + f_{j+1}^n - \frac{1}{\lambda}Q\left(\alpha_{j+1/2}^n\right)\left(w_{j+1}^n - w_j^n\right)\right],$$

with

$$\alpha_{j+1/2} = \lambda\frac{\partial f}{\partial w}\bigg|_{j+1/2} = \begin{cases} \lambda\frac{f_{j+1}-f_j}{w_{j+1}-w_j} & \text{if } w_{j+1}-w_j \neq 0 \\ \lambda\frac{\partial f}{\partial w}\big|_j & \text{if } w_{j+1}-w_j = 0 \end{cases}, \tag{19}$$

and $Q(x)$ is some function, named as the coefficient of numerical viscosity; for the nonhomogeneous case

$$w_t + g(x, w)_x = 0$$

it will be necessary to avoid approximations to the spatial derivatives of the conserved variables, substituting these by discretizations to the spatial derivatives of the flows in order to introduce correctly the source terms, balanced with the fluxes in each volume of control.

For this, we now propose the finite difference scheme in conservation form

$$w_j^{n+1} = w_j^n - \lambda\left[\tilde{g}_{j+1/2} - \tilde{g}_{j-1/2}\right], \tag{20}$$

with a numerical flux $\tilde{g}_{j+1/2}$ defined by

$$\tilde{g}_{j+1/2} = \frac{1}{2}\left[g_j^n + g_{j+1}^n - h\left(\alpha_{j+1/2}^n + \beta_{j+1/2}^n\right)\left(g_{j+1}^n - g_j^n\right)\right] \tag{21}$$

as the generic formulation for a scheme applied to a nonhomogeneous conservation law. In the flux expression, $\alpha_{j+1/2}$ is defined by (19) and, analogously, we denote

$$\beta_{j+1/2} = \begin{cases} \lambda\frac{b_{j+1}-b_j}{w_{j+1}-w_j} & \text{if } w_{j+1}-w_j \neq 0 \\ 0 & \text{if } w_{j+1}-w_j = 0, \end{cases} \tag{22}$$

where

$$b = -\int_0^x s(y, w(y,t))\,dy \quad \text{and} \quad b_i = -\int_0^{x_i} s(y, w(y,t))\,dy.$$

Note that, for the corresponding homogeneous law (18), the characteristic decomposition of the problem can be written as

$$\frac{dw}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = f_w$$

according to the one in which w remains constant along characteristics. In order to find the paths along which w remains constant for the nonhomogeneous case (8), Papalexandris *et al.* [21] propose the following characteristic decomposition

$$\frac{dw}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = f_w - \frac{s}{w_x}.$$

Our estimation for $\alpha_{j+1/2} + \beta_{j+1/2}$ can be interpreted as an averaged numerical expression for this propagation speed multiplied by λ, because

$$\alpha_{j+1/2} + \beta_{j+1/2} \approx \lambda \frac{f_x}{w_x}\bigg|_{j+1/2} + \lambda \frac{b_x}{w_x}\bigg|_{j+1/2} = \lambda\left(f_w - \frac{s}{w_x}\right)\bigg|_{j+1/2}.$$

In the following we can apply well-known results in order to secure the TVD property for the nonhomogeneous case, because the original problem (8) is converted to a homogeneous problem (12). As a result of this analysis, we obtain conditions that $h(x)$ must verify in order to show the TVD property for the scheme described by (20) and (21). For this, we apply the following theorem according to Harten.

THEOREM 2.1. *If a numerical scheme applied to* (18) *is rewritten in the form*

$$w_j^{n+1} = w_j^n + C^+_{j+1/2}\left(w^n_{j+1} - w^n_j\right) - C^-_{j-1/2}\left(w^n_j - w^n_{j-1}\right), \tag{23}$$

where $C^+_{j+1/2}$ and $C^-_{j+1/2}$ are functions of w_j and w_{j+1} which satisfy

$$C^+_{j+1/2} \geq 0, \quad C^-_{j+1/2} \geq 0, \quad \textit{and} \quad C^+_{j+1/2} + C^-_{j+1/2} \leq 1$$

then scheme (23) *is TVD.*

PROPOSITION 2.1. *If $h(x)$ in* (21) *satisfies the following inequalities*

$$\begin{aligned} 1 \leq h(x) \leq \frac{1}{x}, &\quad 0 < x \leq 1 \\ \frac{1}{x} \leq h(x) \leq -1, &\quad -1 \leq x < 0 \end{aligned} \tag{24}$$

then scheme (20) *with the flux defined by* (21) *is TVD under the CFL restriction*

$$\max_j \left|\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right| \leq 1.$$

Proof. Substituting (21) for the numerical flux values in (20),

$$\begin{aligned} w_j^{n+1} = w_j^n - \frac{\lambda}{2}\Big[&\left(g^n_{j+1} - g^n_j\right) + \left(g^n_j - g^n_{j-1}\right) - h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right)\left(g^n_{j+1} - g^n_j\right) \\ &+ h\left(\alpha^n_{j-1/2} + \beta^n_{j-1/2}\right)\left(g^n_j - g^n_{j-1}\right)\Big] \end{aligned}$$

and using (19) and (22), we can rewrite the scheme in the form of (23) with

$$\begin{aligned} C^+_{j+1/2} &= \frac{1}{2}\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right)\left(h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right) - 1\right) \\ C^-_{j+1/2} &= \frac{1}{2}\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right)\left(h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right) + 1\right). \end{aligned}$$

We consider two cases. First, if $0 \le \alpha^n_{j+1/2} + \beta^n_{j+1/2} \le 1$ then

$$h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right) + 1 \ge h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right) - 1 \ge 0$$

and second, if $-1 \le \alpha^n_{j+1/2} + \beta^n_{j+1/2} \le 0$, then

$$h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right) - 1 \le h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right) + 1 \le 0.$$

Therefore, in both cases we have

$$C^+_{j+1/2} \ge 0, \quad C^-_{j+1/2} \ge 0$$
$$C^+_{j+1/2} + C^-_{j+1/2} = \left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right) h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right) \le 1$$

applying the conditions (24) for $h(x)$. Then by Theorem 2.1 the scheme is TVD. ■

PROPOSITION 2.2. *Scheme* (20)–(21) *recognizes steady states for nonhomogeneous problems*, *with at least second-order accuracy.*

Proof. From (21), we have

$$\begin{aligned}\tilde{g}_{j+1/2} - \tilde{g}_{j-1/2} = {} & \frac{1}{2}\left(1 - h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right)\right)\left(g^n_{j+1} - g^n_j\right) \\ & + \frac{1}{2}\left(1 + h\left(\alpha^n_{j-1/2} + \beta^n_{j-1/2}\right)\right)\left(g^n_j - g^n_{j-1}\right).\end{aligned}$$

If the data $\{w^n_j\}$ satisfy the following second-order discretization of the ordinary differential equations' governing steady flow

$$g^n_{j+1} - g^n_j = 0, \quad \forall j \tag{25}$$

then

$$\tilde{g}_{j+1/2} - \tilde{g}_{j-1/2} = 0, \quad \forall j$$

and, from (20), w^{n+1}_j and w^n_j will be equal. Therefore, the scheme (20)–(21) will recognize stationary solutions with at least second-order accuracy. ■

Remark. When the source term vanishes ($g = f$), the modified scheme (20) with the flux defined by (21) has the same generic form as the Harten TVD schemes for some functions depending only on λ and $\alpha_{j+1/2}$. Furthermore, when the speeds associated with the source term can be insignificant as compared to the characteristic speeds; i.e.,

$$\left|\beta_{j+1/2}\right| < \left|\alpha_{j+1/2}\right|, \quad \forall j$$

then

$$\text{sign}\left(\alpha_{j+1/2} + \beta_{j+1/2}\right) = \text{sign}\left(\alpha_{j+1/2}\right)$$

resulting in an upstream differencing scheme with respect to the characteristic associated with the homogeneous case.

Remark. Note that the numerical flux (21), where $h(x) = x$, defines the flux $g^{n+1/2}_{j+1/2}$ for the extension of the Lax–Wendroff scheme. The choice $h(x) = \operatorname{sign}(x)$ leads to an extension of the Courant–Isaacson–Rees scheme. Of course (24), the latter scheme, satisfies the TVD property.

Scheme (20)–(21) with $h(x)$ satisfying the restrictions (24) is only a first-order scheme. The modified equation can be written as

$$w_t + f_x - s = \frac{1}{2}\left[h(\alpha+\beta) - (\alpha+\beta)\right](\Delta x)(f_{xx} - s_x) + O(\Delta x^2) \tag{26}$$

In order to convert the first-order accurate TVD schemes described into second-order accurate ones for the nonhomogeneous case, we use the technique developed in [11] for the homogeneous case. The basic idea is to apply a TVD first-order accurate scheme to the equation

$$w_t + (g+\phi)_x = 0,$$

where g denotes $f - \int_0^x s$ and ϕ is an approximation to the first term of the right-hand side of (26); i.e.,

$$\phi \approx \frac{1}{2}[h(\alpha+\beta) - (\alpha+\beta)](\Delta x) g_x.$$

Hence, to achieve second-order accuracy while retaining the TVD property, we propose the following numerical flux

$$\begin{aligned}\bar{g}_{j+1/2} = \frac{1}{2}\Big[&g^n_j + g^n_{j+1} + \phi^n_j + \phi^n_{j+1} - h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2} + \gamma^n_{j+1/2}\right)\\ &\times \left(g^n_{j+1} - g^n_j + \phi^n_{j+1} - \phi^n_j\right)\Big],\end{aligned}$$

with

$$\gamma_{j+1/2} = \begin{cases} \lambda \frac{\phi_{j+1} - \phi_j}{w_{j+1} - w_j}, & w_{j+1} - w_j \neq 0 \\ 0, & w_{j+1} - w_j = 0 \end{cases}$$

and

$$\phi_j = \begin{cases} s_{j+1/2} \min\left\{\left|\bar{\phi}_{j+1/2}\right|, \left|\bar{\phi}_{j-1/2}\right|\right\}, & \text{if } s_{j+1/2} = s_{j-1/2} \\ 0, & \text{if } s_{j+1/2} \neq s_{j-1/2} \end{cases}, \tag{27}$$

where $s_{j+1/2} = \operatorname{sign}(\bar{\phi}_{j+1/2})$, being

$$\bar{\phi}_{j+1/2} = \frac{1}{2}\left[h\left(\alpha_{j+1/2} + \beta_{j+1/2}\right) - \left(\alpha_{j+1/2} + \beta_{j+1/2}\right)\right](g_{j+1} - g_j). \tag{28}$$

PROPOSITION 2.3. *Let $h(x)$ be such that*

$$
\begin{aligned}
1 \le h(x) \le \frac{1}{x}, &\quad 0 < x \le 1 \\
\frac{1}{x} \le h(x) \le -1, &\quad -1 \le x < 0.
\end{aligned}
\tag{29}
$$

Then the scheme

$$
w_j^{n+1} = w_j^n - \lambda\left[\bar{g}_{j+1/2} - \bar{g}_{j-1/2}\right] \tag{30}
$$

with the flux defined by

$$
\begin{aligned}
\bar{g}_{j+1/2} = {}& \frac{1}{2}\Big[g_j^n + g_{j+1}^n + \phi_j^n + \phi_{j+1}^n - h\left(\alpha_{j+1/2}^n + \beta_{j+1/2}^n + \gamma_{j+1/2}^n\right) \\
& \times \left(g_{j+1}^n - g_j^n + \phi_{j+1}^n - \phi_j^n\right)\Big]
\end{aligned}
\tag{31}
$$

satisfies the following properties:

1. *The scheme is TVD under the CFL restriction*

$$
\max_j \left\{\left|\alpha_{j+1/2}^n + \beta_{j+1/2}^n\right|\right\} \le 1.
$$

2. *Suppose $\Delta t = O(\Delta x)$ and the function $h(x)$ such that $xh(x)$ is Lipschitz continuous*, *then the difference scheme is a second-order accurate scheme*, *except at two consecutive local extreme points.*
3. *The scheme recognizes steady states for the nonhomogeneous problem.*

Proof. We conclude from Theorem 2.1 that the scheme (30)–(31) is TVD under the CFL restriction

$$
\max_j \left\{\left|\alpha_{j+1/2}^n + \beta_{j+1/2}^n + \gamma_{j+1/2}^n\right|\right\} \le 1 \tag{32}
$$

because it can be rewritten as (23) with

$$
\begin{aligned}
C_{j+1/2}^+ &= \frac{1}{2}\left(\alpha_{j+1/2}^n + \beta_{j+1/2}^n + \gamma_{j+1/2}^n\right)\left(h\left(\alpha_{j+1/2}^n + \beta_{j+1/2}^n + \gamma_{j+1/2}^n\right) - 1\right) \ge 0 \\
C_{j+1/2}^- &= \frac{1}{2}\left(\alpha_{j+1/2}^n + \beta_{j+1/2}^n + \gamma_{j+1/2}^n\right)\left(h\left(\alpha_{j+1/2}^n + \beta_{j+1/2}^n + \gamma_{j+1/2}^n\right) + 1\right) \ge 0
\end{aligned}
$$

and

$$
0 \le C_{j+1/2}^+ + C_{j+1/2}^- = \left(\alpha_{j+1/2}^n + \beta_{j+1/2}^n + \gamma_{j+1/2}^n\right)h\left(\alpha_{j+1/2}^n + \beta_{j+1/2}^n + \gamma_{j+1/2}^n\right) \le 1.
$$

Now, notice that ϕ_j^n and ϕ_{j+1}^n cannot change signs without vanishing at the transition point, so that

$$
\left|\phi_{j+1}^n - \phi_j^n\right| \le \max\left\{\left|\phi_{j+1}^n\right|, \left|\phi_j^n\right|\right\} \le \left|\bar{\phi}_{j+1/2}^n\right|
$$

and consequently

$$\left|\gamma^n_{j+1/2}\right| \le \frac{1}{2}\left[\left(\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right)h\left(\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right)-\left(\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right)^2\right]. \tag{33}$$

To derive 1 we show that (32) is implied by the original CFL condition. Note that if the original CFL restriction is satisfied, i.e.,

$$\left|\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right| \le 1,$$

then

$$\begin{aligned}\left|\alpha^n_{j+1/2}+\beta^n_{j+1/2}+\gamma^n_{j+1/2}\right| &\le \left|\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right| + \left|\gamma^n_{j+1/2}\right|\\ &\le \left|\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right| + \frac{1}{2}\left(1-\left(\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right)^2\right)\\ &= 1-\frac{1}{2}\left(\left|\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right|-1\right)^2 \le 1,\end{aligned}$$

using (33) and the inequality $xh(x) \le 1$.

According to Harten, in order to see that (30)–(31) defines a second-order accurate scheme, it is sufficient to show the following relation

$$\bar{g}_{j+1/2} = g^{n+1/2}_{j+1/2} + O(\Delta x^2)$$

for all smooth solutions of (12), because $g^{n+1/2}_{j+1/2}$, described by the equation

$$g^{n+1/2}_{j+1/2} = \frac{1}{2}\left\{g^n_j + g^n_{j+1} - \left(\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right)\left(g^n_{j+1}-g^n_j\right)\right\},$$

defines a numerical flux of a second-order accurate scheme, an extension of the well-known Lax–Wendroff method for a nonhomogeneous conservation law (see [6]). To this end, we obtain

$$\begin{aligned}\bar{g}_{j+1/2} - g^{n+1/2}_{j+1/2} = \frac{1}{2}\Big\{&\phi^n_j + \phi^n_{j+1} + \left[\left(\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right) - h\left(\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right)\right]\left(g^n_{j+1}-g^n_j\right)\\ &- \left[h\left(\alpha^n_{j+1/2}+\beta^n_{j+1/2}+\gamma^n_{j+1/2}\right)\left(g^n_{j+1}-g^n_j+\phi^n_{j+1}-\phi^n_j\right)\right.\\ &\left.- h\left(\alpha^n_{j+1/2}+\beta^n_{j+1/2}\right)\left(g^n_{j+1}-g^n_j\right)\right]\Big\},\end{aligned} \tag{34}$$

where we have added and subtracted the term $h(\alpha^n_{j+1/2}+\beta^n_{j+1/2})(g^n_{j+1}-g^n_j)$ at the right-hand side of (34). First notice that, from (27), the case

$$s^n_{j+1/2} = s^n_{j-1/2}$$

gives

$$\begin{aligned}\phi^n_j &= \frac{1}{2}\left[\bar{\phi}^n_{j+1/2} + \bar{\phi}^n_{j-1/2} - s_{j+1/2}\left|\bar{\phi}^n_{j+1/2} - \bar{\phi}^n_{j-1/2}\right|\right]\\ &= \bar{\phi}^n_{j+1/2} + \frac{1}{2}\left[\bar{\phi}^n_{j-1/2} - \bar{\phi}^n_{j+1/2} - s_{j+1/2}\left|\bar{\phi}^n_{j+1/2} - \bar{\phi}^n_{j-1/2}\right|\right]\\ &= \bar{\phi}^n_{j-1/2} + \frac{1}{2}\left[\bar{\phi}^n_{j+1/2} - \bar{\phi}^n_{j-1/2} - s_{j+1/2}\left|\bar{\phi}^n_{j+1/2} - \bar{\phi}^n_{j-1/2}\right|\right].\end{aligned} \tag{35}$$

If we assume $xh(x)$ is Lipschitz continuous and $\Delta t = O(\Delta x)$, from definition (28), we can conclude

$$\bar{\phi}^n_{j+1/2} - \bar{\phi}^n_{j-1/2} = O(\Delta x^2).$$

Therefore, using the expressions of (35) we have

$$\phi^n_j = \bar{\phi}^n_{j+1/2} + O(\Delta x^2) = \bar{\phi}^n_{j-1/2} + O(\Delta x^2)$$

and consequently

$$\phi^n_j + \phi^n_{j+1} = 2\bar{\phi}^n_{j+1/2} + O(\Delta x^2)$$
$$\phi^n_{j+1} - \phi^n_j = \bar{\phi}^n_{j+1/2} - \bar{\phi}^n_{j-1/2} + O(\Delta x^2) = O(\Delta x^2)$$

or equivalently,

$$\phi^n_j + \phi^n_{j+1} = \left[h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right) - \left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right)\right]\left(g^n_{j+1} - g^n_j\right) + O(\Delta x^2) \quad (36)$$

$$\left|\phi^n_{j+1} - \phi^n_j\right| = O(\Delta x^2) \quad (37)$$

are satisfied. Furthermore, note that if $h(x)$ satisfies the relations (24) then

$$xh(x) - x^2 \geq 0$$

and from the definition of $\bar{\phi}^n_{j+1/2}$ we obtain

$$s^n_{j+1/2} = \operatorname{sign}\left(\bar{\phi}^n_{j+1/2}\right) = \operatorname{sign}\left(w^n_{j+1} - w^n_j\right).$$

Thus, the case $s^n_{j+1/2} \neq s^n_{j-1/2}$ implies

$$w_x|^n_j = 0$$

because w^n_j is a local extreme point. In this case, $\phi^n_j = 0$ but $\bar{\phi}^n_{j+1/2} = O(\Delta x^2)$, then (36) and (37) are also satisfied. We remark that if w^n_j and w^n_{j+1} are two consecutive local extreme points then $\phi^n_j = \phi^n_{j+1} = 0$ and the flux defined by (31) has the same form as (21), describing only a first-order scheme.

Applying (36) we can rewrite (34) in the following form

$$\bar{g}_{j+1/2} - g^{n+1/2}_{j+1/2} = \frac{1}{2}\left[h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right)\left(g^n_{j+1} - g^n_j\right) - h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2} + \gamma^n_{j+1/2}\right)\right.$$
$$\left.\times\left(g^n_{j+1} - g^n_j + \phi^n_{j+1} - \phi^n_j\right)\right] + O(\Delta x^2).$$

Since $xh(x)$ is Lipschitz continuous (we assume the constant to be L), then

$$\left|h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2} + \gamma^n_{j+1/2}\right)\left(g^n_{j+1} - g^n_j + \phi^n_{j+1} - \phi^n_j\right) - h\left(\alpha^n_{j+1/2} + \beta^n_{j+1/2}\right)\left(g^n_{j+1} - g^n_j\right)\right|$$
$$\leq \frac{L}{\lambda}\left|\gamma^n_{j+1/2}\right|\left|w^n_{j+1} - w^n_j\right| = L\left|\phi^n_{j+1} - \phi^n_j\right|$$

and (37) completes the proof of part 2.

Finally, to prove part 3 we consider that the data $\{w_j^n\}$ satisfy the second-order discretization of the ordinary differential equations governing steady flow

$$g_{j+1}^n - g_j^n = 0, \quad \forall j \tag{38}$$

then

$$\bar{\phi}_{j+1/2}^n = 0, \quad \forall j \Rightarrow \phi_j^n = 0, \quad \forall j \Rightarrow \bar{g}_{j+1/2} - \bar{g}_{j-1/2} = 0, \quad \forall j$$

and, from (30),

$$w_j^{n+1} = w_j^n, \quad \forall j$$

with the same accuracy as (38) approximates the stationary equation $f_x = s$. ■

Figure 3 shows the results of applying to the Embid problem the second-order TVD scheme (30)–(31), with $h(\alpha + \beta + \gamma) = \text{sign}\,(\alpha + \beta) + \text{sign}\,(\gamma)$. The time step was chosen such that

$$\max_j \left\{ \left| \alpha_{j+1/2} + \beta_{j+1/2} \right| \right\} = 0.8.$$

The scheme produces an extremely accurate steady solution. The results with the first-order TVD with $h(\alpha + \beta) = \text{sign}\,(\alpha + \beta)$ and the corresponding second-order TVD scheme were both very similar reproducing the exact steady solution except for one internal shock point. The main difference between the two solutions was the convergence rates of both schemes. The first-order TVD scheme required 86 iterations to reach the stationary solution with a residual less than 10^{-10} as compared to the 67 iterations which were needed by the second-order TVD method.

Figure 4 shows the logarithm of residual errors with respect to the number of iterations for both schemes from 30 iterations (the rate of convergence is very similar for the early

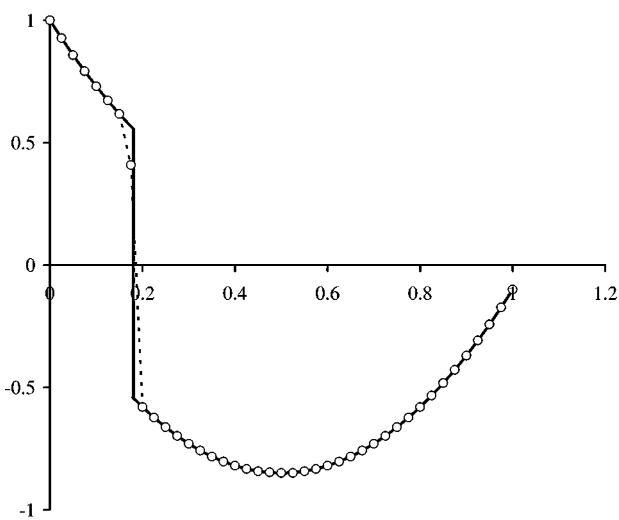

FIG. 3. Steady numerical solution for the Embid problem, calculated with the adapted, second-order TVD scheme, compared with the exact solution (solid line).

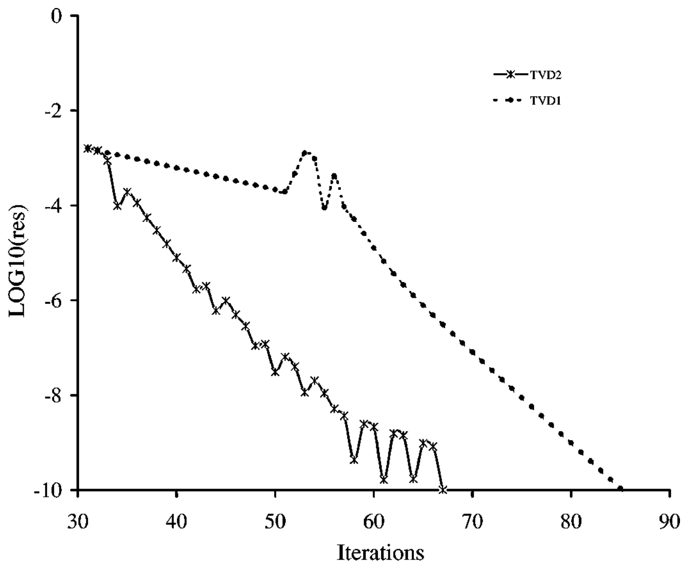

FIG. 4. Convergence histories of the first-order (TVD1) and second-order (TVD2) TVD schemes for the Embid problem.

iterations). Numerical results obtained with the second-order TVD scheme considering $\beta_{j+1/2} = 0$, $\forall j$ lead to similar results but 480 iterations were needed to reach the stationary state.

3. TVD SCHEMES FOR SYSTEMS OF CONSERVATION LAWS WITH SOURCE TERMS

In this section we extend the results of the previous section to hyperbolic systems of conservation laws with source terms of the form

$$W_t + F(W)_x = S(x, W), \tag{39}$$

where $W = W(x, t)$ is the column vector with m components formed by the flow variables, the flux $F(W)$ is a vector-valued function and $S(x, W)$ is the source vector.

Following the method described in the previous section for the scalar case, we can convert the conservation law system with a source term (39) in a homogeneous problem. To this end, we define

$$G(x, W) = F(W) - \int_0^x S(y, W(y, t))\, dy. \tag{40}$$

Hereafter, by simplicity, we will denote

$$B(x, W) = -\int_0^x S(y, W(y, t))\, dy$$

and when the time step is not indicated it will mean that we are considering evaluations in the instant n.

Since $B_x = S$ when $S(., W(., t))$ is a piecewise continuous function, Eq. (39) can be written as

$$W_t + G(x, W)_x = 0.$$

First, we consider the linear, constant coefficient system

$$W_t + JW_x = S, \tag{41}$$

where J is a $m \times m$ constant matrix.

If the system (41) is hyperbolic, the matrix J has real eigenvalues and a complete set of linearly independent right eigenvectors. Let P be the matrix whose columns are the right eigenvectors of J, then

$$J = PDQ, \quad \text{with } Q = P^{-1}, \tag{42}$$

where

$$D = \text{Diag}\,(a_k), \quad k = 1 \ldots m$$

and a_k are the eigenvalues of J.

By choosing a new set of variables U, the characteristic variables, defined by the formula

$$U = QW$$

and multiplying Eq. (41) by Q, we obtain

$$(QW)_t + QJP(QW)_x = QS$$

or

$$U_t + DU_x = QS. \tag{43}$$

This is an uncoupled set of nonhomogeneous scalar equations, which we can solve by applying to each of the m scalar characteristic equations the method described in the previous section for the scalar case. That is,

$$U_j^{n+1} = U_j^n - \lambda\left(\bar{G}^U_{j+1/2} - \bar{G}^U_{j-1/2}\right) \tag{44}$$

with the numerical flux, $\bar{G}^U$, which is defined by

$$\bar{G}^U_{j+1/2} = \frac{1}{2}\left\{G^U_j + G^U_{j+1} + \Phi_j + \Phi_{j+1} - h\left(\lambda \bar{\bar{D}}\right)\left(G^U_{j+1} - G^U_j + \Phi_{j+1} - \Phi_j\right)\right\}, \tag{45}$$

where

$$G^U = QG = Q\left(F - \int_0^x S\right) = DU + QB.$$

Φ_j denotes a m vector, of which components ϕ_j^k are defined by an expression similar to the scalar case, i.e.,

$$\phi_j^k = \begin{cases} s_{j+1/2}^k \min\left\{\left|\bar{\phi}_{j+1/2}^k\right|, \left|\bar{\phi}_{j-1/2}^k\right|\right\}, & \text{if } s_{j+1/2}^k = s_{j-1/2}^k \\ 0, & \text{if } s_{j+1/2}^k \neq s_{j-1/2}^k, \end{cases} \tag{46}$$

where $s_{j+1/2}^k = \operatorname{sign}(\bar{\phi}_{j+1/2}^k)$ and $\bar{\phi}_{j+1/2}^k$ is the component k of the vector $\bar{\Phi}_{j+1/2}$ defined below, together with $h(\lambda\bar{D})$.

If we multiply the expressions (44) and (45) by P to obtain an equation in terms of the original variables, the result is

$$W_j^{n+1} = W_j^n - \lambda\left(\bar{G}_{j+1/2} - \bar{G}_{j-1/2}\right)$$

with

$$\bar{G}_{j+1/2} = \frac{1}{2}\{G_j + G_{j+1} + P(\Phi_j + \Phi_{j+1}) - Ph(\lambda\bar{\bar{D}})[Q(G_{j+1} - G_j) + \Phi_{j+1} - \Phi_j]\}. \tag{47}$$

When J is not a constant matrix, we must choose some average for the matrices $\bar{\bar{D}}_{j+1/2}$, $P_{j+1/2}$, and $Q_{j+1/2}$. In this case, (47) is replaced by

$$\begin{aligned} \bar{G}_{j+1/2} = \frac{1}{2}\big\{&G_j + G_{j+1} + P_{j+1/2}(\Phi_j + \Phi_{j+1}) \\ &- P_{j+1/2}\, h\left(\lambda\bar{\bar{D}}_{j+1/2}\right)\left[Q_{j+1/2}(G_{j+1} - G_j) + \Phi_{j+1} - \Phi_j\right]\big\}, \end{aligned}$$

where Φ_j is defined by (46) from

$$\bar{\Phi}_{j+1/2} = \frac{1}{2}\left(h(\lambda\bar{D})_{j+1/2} - \lambda\bar{D}_{j+1/2}\right) Q_{j+1/2}(F_{j+1} - F_j + B_{j,j+1}).$$

Here we have denoted by $B_{i,j}$ the vector $-\int_{x_i}^{x_j} S(y, W(y, t_n))\, dy$ and

$$\begin{aligned} \lambda\bar{D}_{j+1/2} &= \operatorname{diag}\left(\alpha_{j+1/2}^k + \beta_{j+1/2}^k\right)_{k=1,\ldots,m} \\ h\left(\lambda\bar{D}_{j+1/2}\right) &= \operatorname{diag}\left(h^k\left(\alpha_{j+1/2}^k + \beta_{j+1/2}^k\right)\right)_{k=1,\ldots,m} \\ h\left(\lambda\bar{\bar{D}}_{j+1/2}\right) &= \operatorname{diag}\left(h^k\left(\alpha_{j+1/2}^k + \beta_{j+1/2}^k + \gamma_{j+1/2}^k\right)\right)_{k=1,\ldots,m} \end{aligned}$$

with

$$\alpha_{j+1/2}^k = \lambda\frac{\delta f_{j+1/2}^k}{\delta u_{j+1/2}^k}, \qquad \beta_{j+1/2}^k = \lambda\frac{\delta b_{j+1/2}^k}{\delta u_{j+1/2}^k},$$

where $\delta u_{j+1/2}^k$, $\delta f_{j+1/2}^k$, and $\delta b_{j+1/2}^k$ can also be viewed as the components of the vectors $W_{j+1} - W_j$, $F_{j+1} - F_j$ and $B_{j+1} - B_j$ in the coordinate system $\{P_{j+1/2}^k\}$; i.e., it denotes the component k of

$$Q_{j+1/2}(W_{j+1} - W_j), \quad Q_{j+1/2}(F_{j+1} - F_j), \quad \text{and} \quad Q_{j+1/2}B_{j,j+1},$$

respectively. Finally, $\gamma_{j+1/2}^{k}$ can be calculated as an extension of the scalar case

$$\gamma_{j+1/2}^{k} = \begin{cases} \lambda \frac{\phi_{j+1}^{k} - \phi_{j}^{k}}{\delta u_{j+1/2}^{k}} & \text{if } \delta u_{j+1/2}^{k} \neq 0 \\ 0 & \text{if } \delta u_{j+1/2}^{k} = 0. \end{cases}$$

PROPOSITION 3.1. *The scheme*

$$W_j^{n+1} = W_j^n - \lambda\left(\bar{G}_{j+1/2} - \bar{G}_{j-1/2}\right) \tag{48}$$

with the flux defined by

$$\begin{aligned} \bar{G}_{j+1/2} = \frac{1}{2}\big\{ & G_j + G_{j+1} + P_{j+1/2}(\Phi_j + \Phi_{j+1}) \\ & - P_{j+1/2}\, h\left(\lambda \bar{\bar{D}}_{j+1/2}\right)\left[Q_{j+1/2}(G_{j+1} - G_j) + \Phi_{j+1} - \Phi_j\right]\big\}, \end{aligned} \tag{49}$$

where the diagonal matrix $h(\lambda \bar{\bar{D}}_{j+1/2})$ *is such that* h^k *fulfil the conditions of* (24), *satisfies the following properties*:

1. *When the Jacobian matrix is constant*, *the scheme is TVD under the CFL restriction*

$$\max_k \max_j \left\{\left|\alpha_{j+1/2}^k + \beta_{j+1/2}^k\right|\right\} \leq 1. \tag{50}$$

2. *Suppose* $h^k(x)$ *such that* $x h^k(x)$ *are Lipschitz continuous and* $\Delta t = O(\Delta x)$, *then the difference scheme is a second-order accurate scheme*, *except at two consecutive local extreme points.*
3. *The scheme recognizes steady states for the nonhomogeneous problem.*

Proof. For the Jacobian matrix constant case, we can rewrite the scheme with Eqs. (44) and (45). Now, the scheme is TVD under the CFL restriction (50) by direct application of Proposition 2.3(1) to each characteristic variable. Since

$$G_{j+1/2}^{n+1/2} = \frac{1}{2}\left\{G_j + G_{j+1} - P_{j+1/2}\left(\lambda \bar{D}_{j+1/2}\right) Q_{j+1/2}(G_{j+1} - G_j)\right\}$$

defines the extension of the one-step Lax–Wendroff scheme for nonhomogeneous conservation law systems [6], to show 2, it is sufficient to see that

$$\bar{G}_{j+1/2} = G_{j+1/2}^{n+1/2} + O(\Delta x^2) \tag{51}$$

for all smooth solutions of (39). We can write

$$\begin{aligned} \bar{G}_{j+1/2} - G_{j+1/2}^{n+1/2} = \frac{1}{2} P_{j+1/2}\big\{ & \Phi_j + \Phi_{j+1} + \left[\lambda \bar{D}_{j+1/2} - h\left(\lambda \bar{D}_{j+1/2}\right)\right] Q_{j+1/2}(G_{j+1} - G_j) \\ & - \left[h\left(\lambda \bar{\bar{D}}_{j+1/2}\right)\left(Q_{j+1/2}(G_{j+1} - G_j) + \Phi_{j+1} - \Phi_j\right)\right. \\ & \left. - h\left(\lambda \bar{D}_{j+1/2}\right) Q_{j+1/2}(G_{j+1} - G_j)\right]\big\}, \end{aligned} \tag{52}$$

where we add and subtract the term $P_{j+1/2}h(\lambda\bar{D}_{j+1/2})Q_{j+1/2}(G_{j+1}-G_j)$ at the right-hand side. Likewise, (52) has the form

$$\begin{aligned}\bar{G}_{j+1/2}-G^{n+1/2}_{j+1/2} = \frac{1}{2}\sum_{k=1}^{m} P^k_{j+1/2}\{&\phi^k_j+\phi^k_{j+1}+\left[\left(\alpha^k_{j+1/2}+\beta^k_{j+1/2}\right)\right.\\ &\left.-h^k\left(\alpha^k_{j+1/2}+\beta^k_{j+1/2}\right)\right]\delta g^k_{j+1/2}-\left[h^k\left(\alpha^k_{j+1/2}+\beta^k_{j+1/2}+\lambda\gamma^k_{j+1/2}\right)\right.\\ &\left.\times\left(\delta g^k_{j+1/2}+\phi^k_{j+1}-\phi^k_j\right)-h^k\left(\alpha^k_{j+1/2}+\beta^k_{j+1/2}\right)\delta g^k_{j+1/2}\right]\},\end{aligned}$$

where $\delta g^k_{j+1/2}$ denotes $\delta f^k_{j+1/2}+\delta b^k_{j+1/2}$ and the term enclosed between key brackets are the components of (52) in the coordinate system $\{P^k_{j+1/2}\}$. We conclude the proof of 2 by applying (36) and (37) to each component.

Finally, we notice that replacing (49) in (48), we have

$$\begin{aligned}W^{n+1}_j = W^n_j - \frac{\lambda}{2}\Big\{&(G_{j+1}-G_j)+(G_j-G_{j-1})+P_{j+1/2}(\Phi_j+\Phi_{j+1})\\ &-P_{j-1/2}(\Phi_{j-1}+\Phi_j)-P_{j+1/2}\,h\left(\lambda\bar{\bar{D}}_{j+1/2}\right)\left[Q_{j+1/2}(G_{j+1}-G_j)\right.\\ &\left.+\Phi_{j+1}-\Phi_j\right]+P_{j-1/2}\,h\left(\lambda\bar{\bar{D}}_{j-1/2}\right)\left[Q_{j-1/2}(G_j-G_{j-1})+\Phi_j-\Phi_{j-1}\right]\Big\}. \end{aligned}\quad (53)$$

If $\{W^n_j\}$ approximates the stationary solution associated with the nonhomogeneous conservation law system, we have

$$G_{j+1}-G_j=0,\quad \forall j \quad (54)$$

with at least a second-order accuracy. Then, using the corresponding definitions for $\bar{\Phi}_{j+1/2}$ and Φ_j,

$$\bar{\Phi}_{j+1/2}=0,\quad \forall j \Rightarrow \Phi_j=0,\quad \forall j$$

and, from (53),

$$W^{n+1}_j=W^n_j,\quad \forall j$$

with the same accuracy as (54) approximates the steady equation $F_x=S$. ■

In order to implement scheme (48)–(49), we notice that we need an approximation of integrals of the form $\int_0^{x_j} S(y,W(y,t_n))\,dy\ \forall j$, according to (40). In this case, it might be advantageous to use independent integrals over each control volume. To this end, we observe that substracting B_j from $\bar{G}_{j+1/2}$ and $\bar{G}_{j-1/2}$ leads to the same scheme and we can write

$$\begin{aligned}\frac{1}{2}\{G_j+G_{j+1}\}-B_j &= \frac{1}{2}\left\{F_j+F_{j+1}-B_{j,j+1/2}+B_{j+1/2,j+1}\right\}+B_{j,j+1/2}\\ \frac{1}{2}\{G_{j-1}+G_j\}-B_j &= \frac{1}{2}\left\{F_j+F_{j+1}-B_{j-1,j-1/2}+B_{j-1/2,j}\right\}-B_{j-1/2,j}.\end{aligned}$$

It can be verified that, with these simplifications, our numerical second-order TVD method (48)–(49) takes the form

$$W^{n+1}_j=W^n_j-\lambda\left[G^{TVD2}_{j+1/2}-G^{TVD2}_{j-1/2}\right]-\lambda\left[B_{j-1/2,j}+B_{j,j+1/2}\right], \quad (55)$$

where

$$G^{TVD2}_{j+1/2} = \frac{1}{2}\big\{F_j + F_{j+1} - B_{j,j+1/2} + B_{j+1/2,j+1} + P_{j+1/2}(\Phi_j + \Phi_{j+1}) \\ - P_{j+1/2}\, h\big(\lambda \bar{\bar{D}}_{j+1/2}\big)\big[Q_{j+1/2}(F_{j+1} - F_j + B_{j,j+1}) + \Phi_{j+1} - \Phi_j\big]\big\}. \quad (56)$$

Analogously, when $\Phi_j = 0\ \forall j$, the above equations describe first-order TVD schemes which can be written as

$$W^{n+1}_j = W^n_j - \lambda\big[G^{TVD1}_{j+1/2} - G^{TVD1}_{j-1/2}\big] - \lambda\big[B_{j-1/2,j} + B_{j,j+1/2}\big], \quad (57)$$

where

$$G^{TVD1}_{j+1/2} = \frac{1}{2}\big\{F_j + F_{j+1} - B_{j,j+1/2} + B_{j+1/2,j+1} \\ - P_{j+1/2}\, h\big(\lambda \bar{D}_{j+1/2}\big)\, Q_{j+1/2}(F_{j+1} - F_j + B_{j,j+1})\big\}. \quad (58)$$

Remark. The weight of the numerical source terms in the complete first-order TVD scheme (57)–(58) can be written by

$$\frac{1}{2}\big[I + P_{j-1/2}\, h\big(\lambda \bar{D}_{j-1/2}\big)\, Q_{j-1/2}\big]\, B_{j-1,j} + \frac{1}{2}\big[I - P_{j+1/2}\, h\big(\lambda \bar{D}_{j+1/2}\big)\, Q_{j+1/2}\big]\, B_{j,j+1}, \quad (59)$$

which takes the same form as the approximation suggested by Bermúdez and Vázquez in [2] for the source term. For the particular case in which $B_{j,j+1}$ is evaluated applying the rectangular or the trapezoidal rule from the x_j and x_{j+1} nodes, and $h(\lambda \bar{D}_{j+1/2})$ in (59) is chosen as

$$\operatorname{diag}\big(\operatorname{sign}\big(\alpha^k_{j+1/2}\big)\big)$$

we obtain, according to the choice of the Jacobian matrix average, the extensions of the Q-schemes of Roe and van Leer presented in [2] for nonhomogeneous conservation laws. To this end, we have taken into account that

$$\operatorname{sign}(x)x = |x|.$$

4. GOVERNING EQUATIONS

In order to show the efficiency and accuracy of the described TVD schemes for nonhomogeneous hyperbolic systems, quasi-one-dimensional nozzle flows are used. The governing equations for the quasi-one-dimensional unsteady flow through a duct of varying cross-section can be written in conservation form as

$$W_t + F(W)_x = S(x, W), \quad (60)$$

where

$$W(x,t) = \left(\rho A,\ \rho u A,\ \left(\rho\frac{u^2}{2} + \frac{p}{\gamma - 1}\right) A\right)^T$$

$$F(W) = \left(\rho u A, (\rho u^2 + p)A, u\left(\rho\frac{u^2}{2} + p\frac{\gamma}{\gamma - 1}\right)A\right)^T$$

$$S(x, W) = (0, pA'(x) - g\rho A_l, q\rho A_l)^T.$$

The system is closed by the state equation $e = \rho\frac{u^2}{2} + \frac{p}{\gamma - 1}$.

Here, the quantities $\rho, u, p,$ and e represent the density, velocity, pressure, and total energy; A is the cross-section, γ denotes the ratio of specific heat capacities of the gas, q is the heat transfer energy per unit mass per unit time, g is the friction term, and A_l corresponds to the wall surface per unit of length (in this case, it is the value of the duct diameter).

The source term vector $S(x, W)$ includes terms related to the variation of pipe cross-sectional area A and dissipative terms related to the presence of friction forces and heat transfer at pipe walls, which render the flow nonhomentropic.

The Jacobian matrix of the flux function is given by

$$J = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma - 3}{2}u^2 & (3 - \gamma)u & \gamma - 1 \\ u\left(\frac{\gamma - 1}{2}u^2 - H\right) & H - (\gamma - 1)u^2 & \gamma u \end{pmatrix}, \tag{61}$$

where $H = \frac{a^2}{\gamma - 1} + \frac{u^2}{2}$ is the entalphy and a denotes the sound speed. The Jacobian matrix is hyperbolic and has three eigenvalues

$$u - a, \quad u, \quad u + a.$$

The corresponding right eigenvectors form the matrix $P = (P^1, P^2, P^3)$, with

$$P^1 = \begin{pmatrix} 1 \\ u - a \\ H - ua \end{pmatrix}, \quad P^2 = \begin{pmatrix} 1 \\ u \\ \frac{u^2}{2} \end{pmatrix}, \quad P^3 = \begin{pmatrix} 1 \\ u + a \\ H + ua \end{pmatrix}.$$

That is to say, the matrix P diagonalizes J so that

$$P^{-1}JP = \mathrm{diag}(u - a, u, u + a),$$

where P^{-1} is formed by the corresponding left eigenvectors matrix. For simplicity, we denote P^{-1} as Q. Then $Q = (Q_1, Q_2, Q_3)^T$, where

$$Q_1 = \left(\frac{u}{2a} + \frac{\gamma - 1}{4}\frac{u^2}{a^2}, -\frac{1}{2a} - \frac{\gamma - 1}{2}\frac{u}{a^2}, \frac{\gamma - 1}{2a^2}\right)$$

$$Q_2 = \left(1 - \frac{\gamma - 1}{2}\frac{u^2}{a^2}, (\gamma - 1)\frac{u}{a^2}, \frac{1 - \gamma}{a^2}\right)$$

$$Q_3 = \left(-\frac{u}{2a} + \frac{\gamma - 1}{4}\frac{u^2}{a^2}, \frac{1}{2a} - \frac{\gamma - 1}{2}\frac{u}{a^2}, \frac{\gamma - 1}{2a^2}\right).$$

5. NUMERICAL RESULTS

In this section we show computations using two test problems to demonstrate the performance of the previously described methods for solving (60). All numerical results were computed by the following particular second-order TVD scheme

$$W_j^{n+1} = W_j^n - \lambda\left[G_{j+1/2}^{MTVD} - G_{j-1/2}^{MTVD}\right] - \lambda\left[B_{j-1/2,j} + B_{j,j+1/2}\right], \tag{62}$$

where

$$\begin{aligned} G_{j+1/2}^{MTVD} = \frac{1}{2}\{&F_j + F_{j+1} - B_{j,j+1/2} + B_{j+1/2,j+1} - P_{j+1/2}\, h\left(\lambda\bar{D}_{j+1/2}\right) \\ &\times Q_{j+1/2}(F_{j+1} - F_j + B_{j,j+1})\} + P_{j+1/2}\Psi_{j+1/2} \end{aligned} \tag{63}$$

and $\Psi_{j+1/2}$ is the vector whose components are given by

$$\varphi_{j+1/2}^k = s_{j+1/2}^k \max\left\{0, \min\left\{\left|\bar{\phi}_{j+1/2}^k\right|, s_{j+1/2}^k\bar{\phi}_{j-1/2}^k, s_{j+1/2}^k\bar{\phi}_{j+3/2}^k\right\}\right\}$$

representing the second-order contribution of the scheme. The calculations were performed choosing the matrix h as

$$h\left(\lambda\bar{D}_{j+1/2}\right) = \operatorname{diag}\left(\operatorname{sign}\left(\alpha_{j+1/2}^k + \beta_{j+1/2}^k\right)\right)_{k=1,2,3}.$$

The scheme (62)–(63) is the particular case of Eqs. (55) and (56) when $h(\lambda\bar{\bar{D}}_{j+1/2})$ is defined as

$$h\left(\lambda\bar{\bar{D}}_{j+1/2}\right) = \operatorname{diag}\left(\frac{\left|\alpha_{j+1/2}^k + \beta_{j+1/2}^k\right| + \left|\gamma_{j+1/2}^k\right|}{\alpha_{j+1/2}^k + \beta_{j+1/2}^k + \gamma_{j+1/2}^k}\right)_{k=1,2,3}.$$

For the system governing the quasi-one-dimensional unsteady flow through a duct of varying cross section, the α^k are obtained by

$$\alpha^1 = \lambda(u-a), \quad \alpha^2 = \lambda u, \quad \alpha^3 = \lambda(u+a)$$

and for β^k we have the following expressions

$$\beta^1 = \lambda\frac{\left(\frac{1}{2a} + \frac{\gamma-1}{2}\frac{u}{a^2}\right)(pA'(x) - g\rho A_l) - \frac{\gamma-1}{2a^2}(q\rho A_l)}{-\frac{1}{2a}\rho A\frac{\partial u}{\partial x} + \frac{1}{2a^2}\frac{\partial(pA)}{\partial x}}$$

$$\beta^2 = \lambda\frac{(1-\gamma)\frac{u}{a^2}(pA'(x) - g\rho A_l) - \frac{1-\gamma}{a^2}(q\rho A_l)}{\left(\frac{\partial(\rho A)}{\partial x} - \frac{1}{a^2}\frac{\partial(pA)}{\partial x}\right)}$$

$$\beta^3 = \lambda\frac{\left(-\frac{1}{2a} + \frac{\gamma-1}{2}\frac{u}{a^2}\right)(pA'(x) - g\rho A_l) - \frac{\gamma-1}{2a^2}(q\rho A_l)}{\frac{1}{2a}\rho A\frac{\partial u}{\partial x} + \frac{1}{2a^2}\frac{\partial(pA)}{\partial x}}.$$

In the event that a denominator in the expressions for β^k becomes zero or very small, we consider $\beta^k = 0$ and the propagation speeds are the corresponding characteristic speeds associated with the homogeneous flow.

In order to implement the difference finite scheme described by (62) and (63) to find an approximation for the solution of the system (60) with initial conditions, we use Roe's linearization technique to obtain an average in the middle for u, H, ρ, and ρA. For our system of equations, this averaging takes the following form:

$$\chi_{j+1/2} = \sqrt{\frac{\rho_{j+1}A_{j+1}}{\rho_j A_j}}, \quad u_{j+1/2} = \frac{\chi_{j+1/2}u_{j+1} + u_j}{\chi_{j+1/2}+1}, \quad H_{j+1/2} = \frac{\chi_{j+1/2}H_{j+1} + H_j}{\chi_{j+1/2}+1},$$
$$\rho_{j+1/2} = \sqrt{\rho_j \rho_{j+1}}, \quad \rho_{j+1/2}A_{j+1/2} = \sqrt{\rho_j A_j \rho_{j+1} A_{j+1}}.$$

Following these averages, a natural approximation of the matrices P, $h(\lambda\overline{D})$, and Q in the middle can be obtained. Regarding the $\beta^k_{j+1/2}$ estimations, we use a first-order approximation for the partial derivatives. Additionally, to estimate the source terms $B_{j,j+1}$ we propose the following evaluations

$$B_{j,j+1} = \left(0, -p_{j+1/2}(A_{j+1} - A_j) + (\Delta x)(g\rho A)_{j+1/2}, -(\Delta x)(q\rho A_l)_{j+1/2}\right)^T.$$

In the particular context of the quasi-one-dimensional unsteady flow through a duct of varying cross-section, it is important to distinguish between the approximations used for the term related to the variation of the pipe cross-sectional area, $pA'(x)$, and these ones used to approximate the terms related to the presence of friction forces and heat transfer at pipe walls, which are less important.

We note that $pA'(x)$ should always be balancing the second component of the fluxes, therefore treated as a divergence term, as it really is because it represents the divergence of the surface forces over the lateral walls of the control volume. Because of this, it is important to approximate the integral of $pA'(x)$ between x_j and x_{j+1} as an expression of the form

$$p_{j+1/2}(A_{j+1} - A_j), \tag{64}$$

where $p_{j+1/2}$ represents some average of the pressure between x_j and x_{j+1}.

Note that if we consider the particular case in which at time t_n

$$\rho^n_j = \bar{\rho}, \quad p^n_j = \bar{p}, \quad \text{and} \quad u^n_j = 0, \quad \forall j,$$

where $\bar{p}$ and $\bar{\rho}$ are constants, then

$$G_{j+1} - G_j \underset{\substack{\uparrow \\ u_j = 0}}{=} \left(\begin{array}{c} 0 \\ p_{j+1}A_{j+1} - p_j A_j - p_{j+1/2}(A_{j+1} - A_j) \\ 0 \end{array} \right) \underset{\substack{\uparrow \\ p_j = \bar{p}}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall j$$

and using the proof of Proposition 3.1 (part 3), we have

$$W^{n+1}_j = W^n_j, \quad \forall j$$

consequently

$$\rho^{n+1}_j = \rho^n_j = \bar{\rho}, \quad p^{n+1}_j = p^n_j = \bar{p}, \quad \text{and} \quad u^{n+1}_j = 0, \quad \forall j$$

which is the exact physical solution.

Since

$$\rho(x,t) = \bar{\rho}, \quad p(x,t) = \bar{p}, \quad \text{and} \quad u(x,t) = 0 \tag{65}$$

is a stationary solution to (60), which in this case can be written by

$$\frac{\partial}{\partial x}(pA) = pA'(x), \tag{66}$$

the above analysis can be interpreted, in the context of the $\mathcal{C}$*-property* introduced by Bermúdez and Vázquez in [2], as that the scheme (62)–(63) with the discretization defined by (64) for the source terms, satisfies the *exact* $\mathcal{C}$*-property* relating to the stationary solution (65) when applied to the stationary problem given by (66).

Analogously, if the expression (64) in $B_{j,j+1}$ is replaced by an approximation of the form

$$p_{j+1/2} A'\left(x_{j+1/2}\right) \Delta x$$

then the scheme (62)–(63) only satisfies the *approximate* $\mathcal{C}$*-property* relating to the stationary solution (65) for Problem (66).

With respect to the averages used to evaluate the pressure and the others quantities (u, H, ρ, and ρA) in the approximation of the vector $B_{j,j+1}$, the differences are less obvious. By example, our experiments indicate that replacing Roe's average by the simple arithmetic average leads to similar results.

Two convergent–divergent nozzles with different geometry are chosen for our experiments. In both cases, the problem is outlined as the calculation of nonsteady compressible flow which is established between two atmospheres, connected by a nozzle, when initially the ideal separation existing between the atmospheres and the pipe at both ends is instantaneously removed. In all the cases, we show the numerical results when the steady solution is reached. The calculation of the analytical steady solutions for these classical problems can be found in [1] and [14], for example.

5.1. *Problem* 1

The first problem, proposed by Anderson in [1], is concerned with a convergent–divergent nozzle with a parabolic area distribution given by

$$A(x) = 1 + 2.2(x - 1.5)^2, \quad 0 \le x \le 3. \tag{67}$$

This nozzle is illustrated in Fig. 5.

The calculations have been performed with 1 bar of pressure at the left side and 0.6784 at the right and 300 K of temperature at both sides, in such a way that a shock is established inside the pipe.

All the calculations were performed with CFL equal to 0.9 and uniform computational grid inside the nozzle. A half length mesh at both ends of the pipe was used. This change avoids the conservation mistakes ocurring along the pipe as a consequence of the mismatching between the calculation of the flow variables at both ends of the pipe and the interior region (see [5]).

In Figs. 6 and 7 we show the numerical results (circles) of pressure and Mach number obtained by the second-order TVD scheme (62)–(63) when the steady solution is carried out.

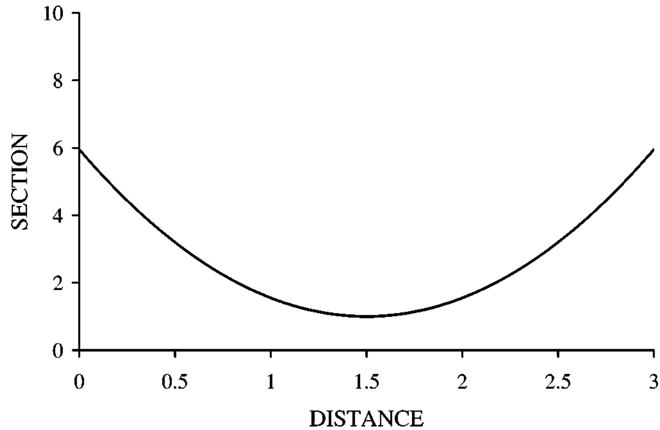

FIG. 5. Convergent–divergent nozzle for Problem 1.

The calculations were performed when $t = 0.1$ s and using 51 points. To see the accuracy of the TVD scheme, the computed results are compared with the exact solutions (solid line).

In this case, and due to the strong influence of the throat on the flow, it has not been possible to obtain any solution with the classic, two-step Lax–Wendroff method, a natural extension for the vectorial case of Eqs. (9) with (10). The method failed because of a nonphysical overshoot which leads to negative thermodynamic conditions at some point in the pipe. The extension for the system case of the scheme described by (16) and (17), named one-step, adapted Lax–Wendroff scheme, allows for a solution. However, the accuracy of the obtained results is, in this case, quite poor. As is observed in Fig. 8, a strong false shock (subsonic–supersonic) appears at the throat, spoiling the solution. This is not only due to the oscillatory nature of the adapted Lax–Wendroff method, but also because this method has no control over the satisfaction of the positive entropy variation requirement.

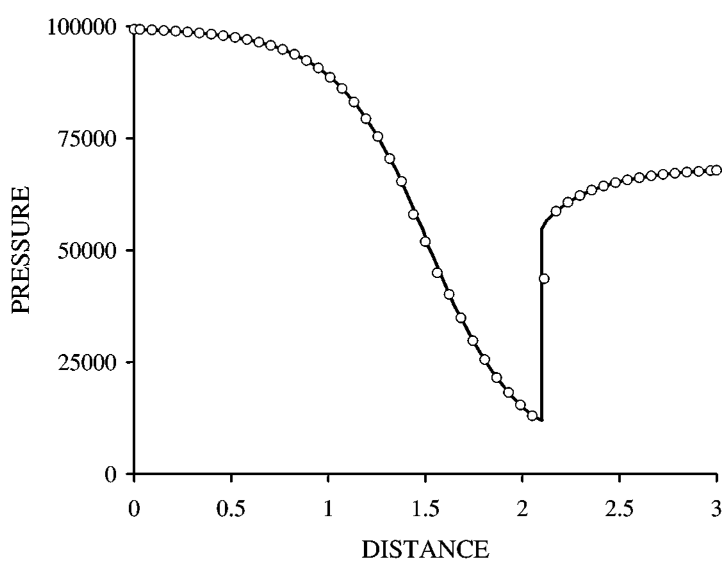

FIG. 6. Pressure steady results for the nozzle of Fig. 5, calculated with scheme (62)–(63) using 51 nodes.

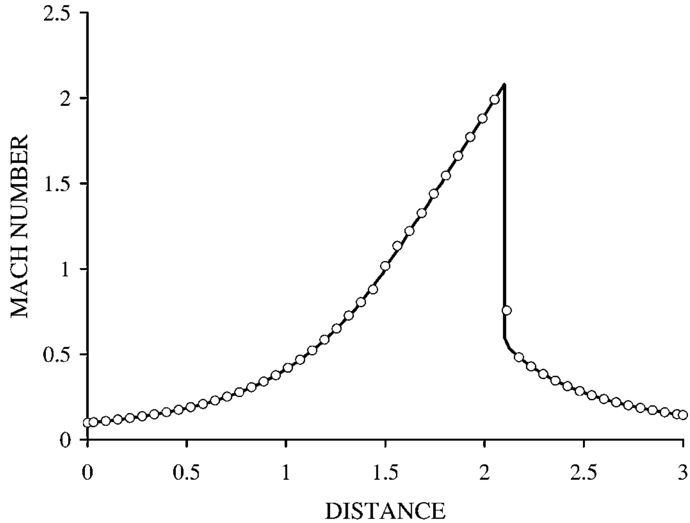

FIG. 7. Mach number steady results for the nozzle of Fig. 5, calculated with scheme (62)–(63) using 51 nodes.

In order to obtain a good solution to this kind of problem, one in which the cross-section variation has a strong influence on the flow due to the existence of supersonic velocities and shocks, it is essential to use a nonoscillatory technique. The numerical solution obtained with the TVD scheme shows very good conservation and gives a reasonable resolution of the shock even for a lower number of nodes as can be seen in Figs. 9 and 10, which represent the pressure and Mach number steady solutions obtained with a 25-uniform grid.

5.2. *Problem* 2

The flow through the convergent–divergent nozzle represented in Fig. 11 is the second selected test problem. It has been proposed by the authors in [4] as a more difficult

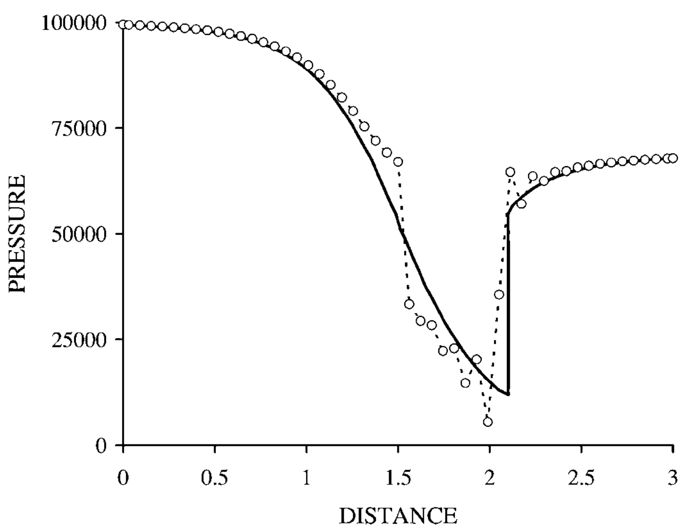

FIG. 8. Pressure steady results obtained with the adapted, one-step Lax–Wendroff method for the nozzle of Fig. 5 (51 nodes).

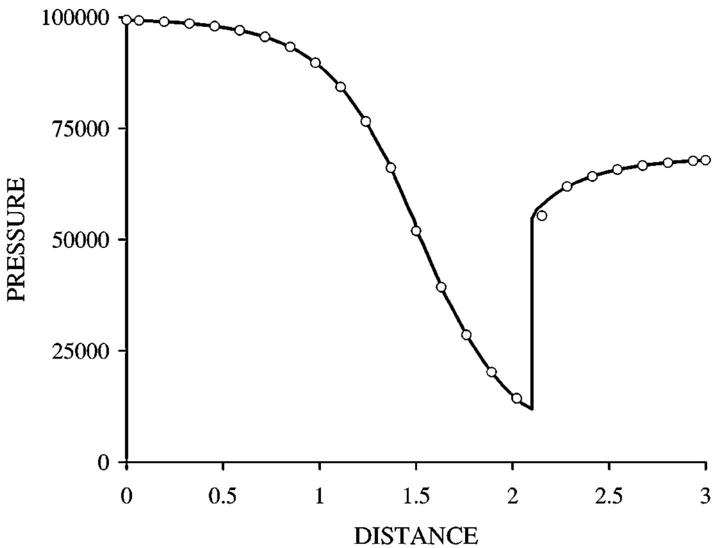

FIG. 9. Pressure steady results for the nozzle of Fig. 5, calculated with scheme (62)–(63) using 25 nodes.

test because of the existence of a discontinuity for $A'(x)$ at the throat. A pipe with a 1-m length, 0.05-m diameter at both sides, and 0.038-m diameter at the throat has been chosen.

For this problem, the flow is defined as the homentropic release of pure air through the indicated nozzle, from the left atmosphere at 2 bars of pressure and 300 K of temperature, to the right atmosphere at 1.5 bars of pressure.

Figures 12 and 13 show the pressure and the velocity steady numerical solutions for this problem calculated with the second-order TVD scheme (62)–(63) using 51 points compared

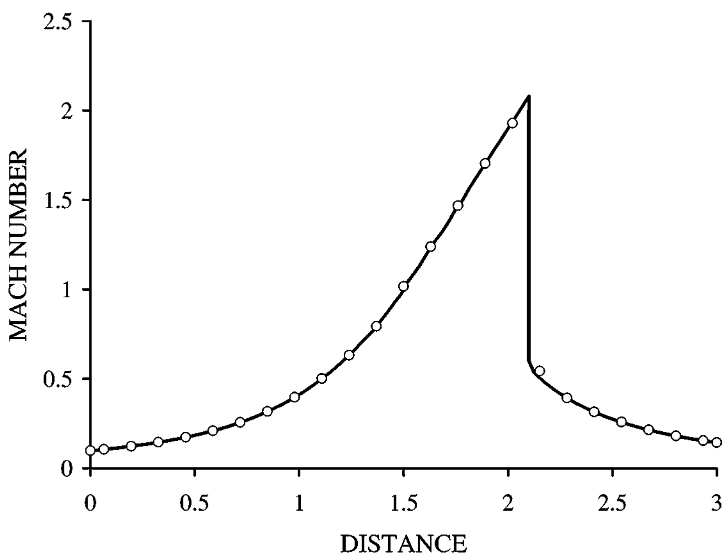

FIG. 10. Mach number steady results for the nozzle of Fig. 5, calculated with scheme (62)–(63) using 25 nodes.

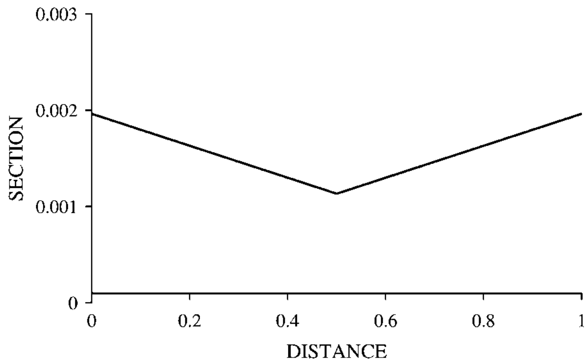

FIG. 11. Convergent–divergent nozzle for Problem 2.

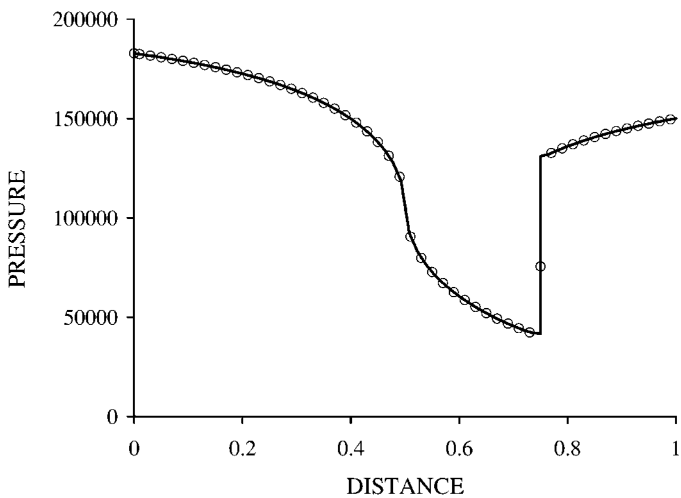

FIG. 12. Pressure steady results for the convergent–divergent nozzle of Fig. 11, calculated with scheme (62)–(63) using 51 nodes.

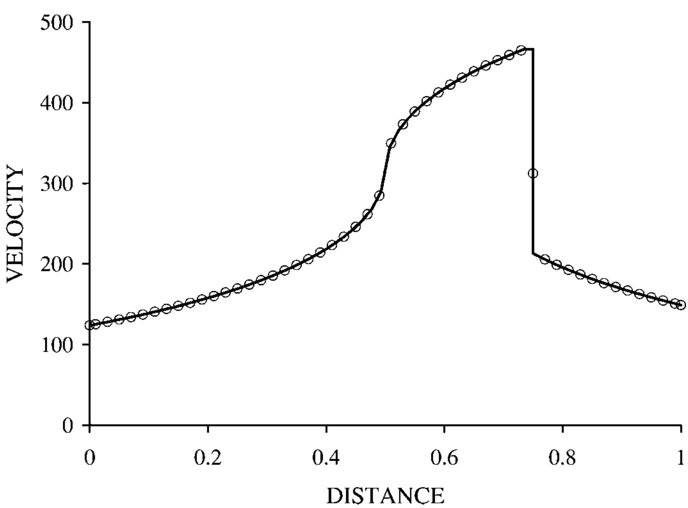

FIG. 13. Velocity steady results for the nozzle of Fig. 11, calculated with scheme (62)–(63) using 51 nodes.

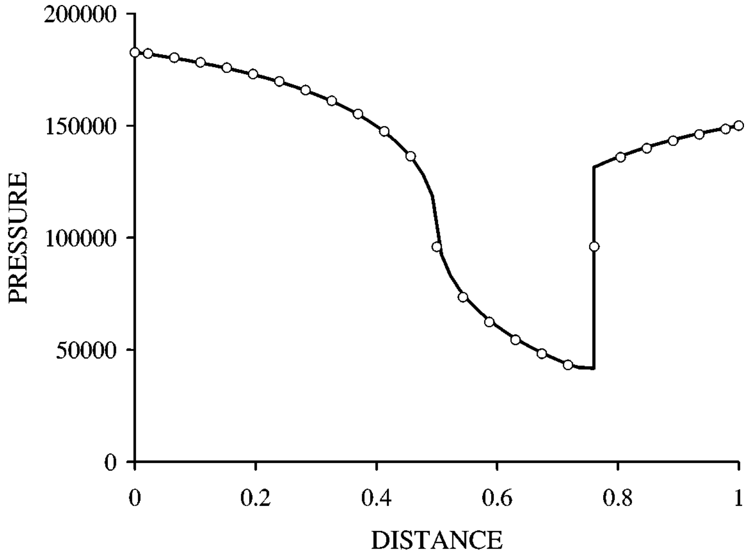

FIG. 14. Pressure steady results for the nozzle of Fig. 11, calculated with scheme (62)–(63) using 25 nodes.

with the exact solution (solid line). Analogously, Figs. 14 and 15 show the pressure and the velocity steady numerical solutions calculated using 25 mesh points.

As observed, with the second-order TVD scheme the solution is very close to the exact solution and no false shock is obtained. A wrong point, the internal shock point, is observed in the pressure and velocity numerical solutions calculated with the TVD scheme. This is due to the fact that, at the control volume, in which the shock is located, the flow properties are space averaged values between the corresponding subsonic and supersonic states and therefore it leads to a nonreal physical solution at that point.

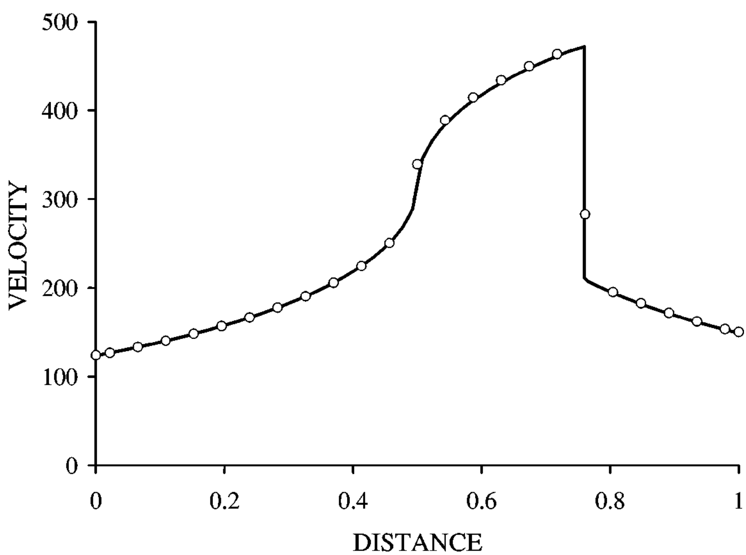

FIG. 15. Velocity steady results for the convergent–divergent nozzle of Fig. 11, calculated with scheme (62)–(63) using 25 nodes.

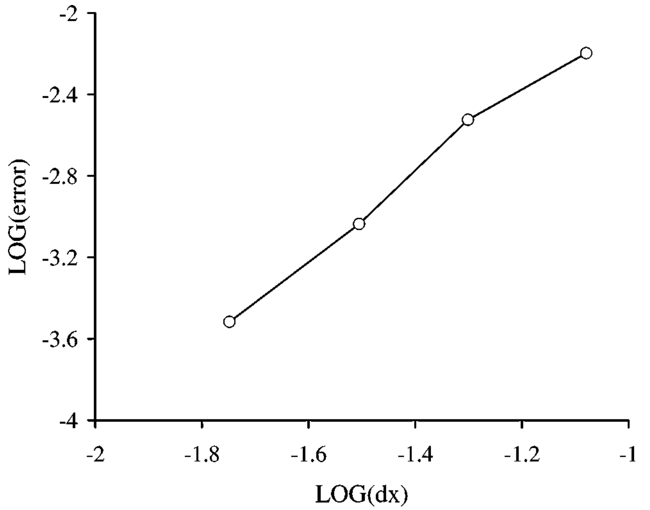

FIG. 16. Error against mesh size of the scheme (62)–(63) for a shock-free test case in the convergent–divergent nozzle of Fig. 11.

In order to illustrate the accuracy of the scheme (62)–(63), a shock-free test problem for the nozzle of Fig. 11 has been considered. In this case, the pressure at the right atmosphere is chosen to be 1.9 bars. Using as reference the mass flow-rate solution, which is constant at steady state, the errors measured by root-mean-square for different mesh sizes have been plotted in Fig. 16. These indicate that the proposed TVD scheme is second-order accurate at the steady state. Although results are omitted, similar conclusions were obtained using the corresponding first-order TVD scheme; i.e., both first- and second-order TVD methods constructed in the present research approximate steady solutions with second-order accuracy.

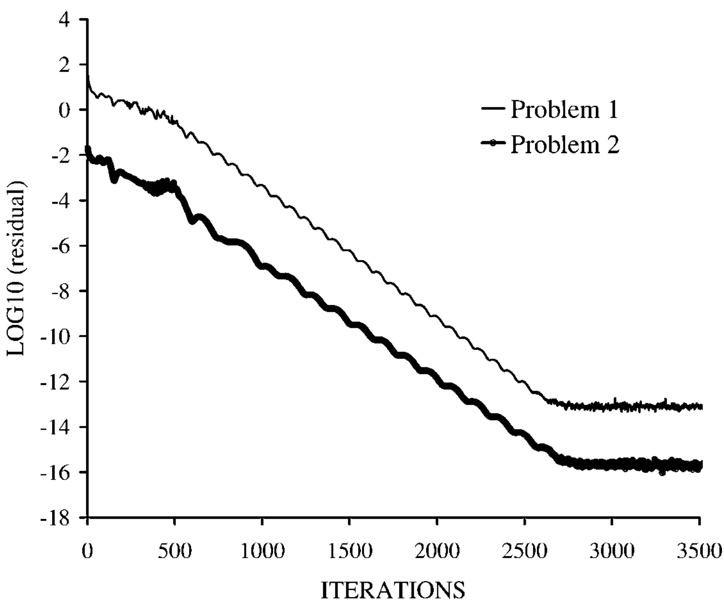

FIG. 17. Convergence histories of scheme (62)–(63) for the mass flow rate solution in Problems 1 and 2.

Figure 17 shows the convergence history for the mass flow-rate solution obtained with the TVD scheme (62)–(63) for Problems 1 and 2, respectively. As can be seen, the speed of convergence to the steady state is very similar in both cases. It is important to note that we have not found differences between the convergence histories achieved with this scheme and with the one of first-order.

Although the TVD scheme described by (62)–(63) provides a solution very close to the exact solution for the problems described in this paper, the method also may admit entropy violating discontinuities as solution. Note that when the source term vanishes the corresponding first-order method defines the Roe's scheme, which is an entropy violating scheme. In [4], we have proposed a modification of this method in order to force the satisfaction of the entropy condition.

6. CONCLUSIONS

A general technique to construct numerical methods with a capacity to recognize steady solutions for hyperbolic conservation law systems with source terms has been presented. We propose the transformation of the nonhomogeneous conservation law problem into a homogeneous one, introducing a new flux which is generated by adding the primitive of the source term to the physical flux.

The technique developed in this paper can be seen as a formalization of the empirical method suggested by Roe in [23] for the inclusion of source terms in a general high-order scheme. This technique is more general and can be applied to extend well-known schemes to nonhomogeneous conservation laws, guaranteeing the balance of the flux and source terms at steady states. Nevertheless, to obtain the mentioned balance, the formulation of the schemes must be such that all the differences that appear in the schemes are expressed as flux-differences including source terms and not as flow variable differences, which prevent the application of the some classic schemes from being immediate, being in some instances impossible as is the case of the two-step Lax–Wendroff method, for example.

In this paper we have dealt with the generalization to nonhomogeneous conservation laws of the explicit, second-order TVD schemes introduced by Harten in [11]. The extensions of the Q-schemes of Roe (originally introduced in [23]) and van Leer, proposed both in [2], are obtained as particular cases of first-order TVD schemes. Extensions of others flux-limiters and flux-vector splitting techniques for nonhomogeneous hyperbolic conservation laws, following the described technique, is the subject of work presently in progress.

The developed schemes have been applied to the calculation of quasi-one-dimensional flow through pipes with a variable cross-section. In these problems the variation of the cross-section, included in the source term, has a strong influence on the equations and therefore said problems are an excellent reference to test the stated theory. The second-order TVD schemes were found to be robust and with the ability to capture steady solutions accurately. These schemes can also be applied to other hyperbolic conservation laws with source terms, e.g., the shallow water.

Although the interest of this paper has been on the development of high-order numerical techniques with a capacity to recognize steady states of nonhomogeneous hyperbolic conservation laws, the proposed schemes were tested on a scalar conservation law with a stiff source term. Numerical experiments confirmed that the introduction of the new numerical speeds associated with the source term gives the correct propagation speeds of discontinuities for an acceptable level of stiffness.

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